RENORMALIZATION GROUP ANALYSIS OF NONLINEAR DIFFUSION EQUATIONS WITH TIME DEPENDENT COEFFICIENTS: ANALYTICAL RESULTS

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Abstract

We study the long-time asymptotics of a certain class of nonlinear diffusion equations with time-dependent diffusion coefficients which arise, for instance, in the study of transport by randomly fluctuating velocity fields. Our primary goal is to understand the interplay between anomalous diffusion and nonlinearity in determining the long-time behavior of solutions. The analysis employs the renormalization group method to establish the self-similarity and to uncover universality in the way solutions decay to zero.

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1 Introduction

Theories of transport by a random velocity field are used in a number of important problems in many fields of science and engineering. Examples range from mass and heat transport in geophysical flows [14], to combustion and chemical engineering [24], to hydrology [13] and petroleum engineering [20]. In each of these examples one finds important physical processes which involve the transport of a passive scalar quantity in the presence of a complex velocity field that fluctuates randomly on length and time scales comparable to those on which the transport process occurs. A central goal of transport theories is to understand the effects produced by such random velocity fluctuations on the mean (ensemble averaged) transport.

A characteristic feature of these theories, which are usually based on perturbation methods, are infrared (long wavelength or low frequency) divergences in the terms of the pertinent perturbation expansions due to long-range (nonintegrable) spatial or temporal correlations in velocity fluctuations. See, e.g., [2, 16, 22] and references therein. Typically, the dominant divergences in lowest order are of diffusion type, and correspond to anomalous diffusion in the ensemble averaged transport equations, for which the effective diffusion coefficient increases with time and is divergent as time $t \to \infty$. So, according to these theories the mean concentration, u, of a passive scalar field being advected by a random velocity field with strong, long-range correlations satisfies, under appropriate conditions, an equation of the form [16, 22]

$$u_t = c(t)u_{xx} + F(u, u_x, u_{xx}), \quad c(t) \sim t^p \text{ as } t \to \infty, \text{ with } p > 0. \tag{1}$$

It is our intent here and in [6] to analyze the long-time behavior of solutions of equations of the form (1). The analysis assumes $F = F(u)$ to be superlinear, in the sense that $F(u) = O(u^\alpha)$ as $u \to 0$, with $\alpha > 1$. The inclusion of the nonlinear term $F(u)$ in (1) accounts for situations in which the scalar field is not conservative, meaning that its concentration u undergoes changes due to physical, chemical or

biological processes.

We are concerned primarily with the interplay between anomalous diffusion (measured in terms of the exponent p) and nonlinearity (measured in terms of the exponent α) in determining the scaling behavior of solutions as they decay to zero. In the present paper we analyze the situation where the nonlinearity is analytic and "supercritical" (or irrelevant), in the sense that $F(u) = \sum_{j\geq\alpha} a_j u^j$, with $\alpha > (p+3)/(p+1)$. We show that in this case diffusion is the dominant effect in the limit $t \to \infty$, and determines the scaling form of solutions with sufficiently localized initial data as they decay to zero:

$$u(x,t) \sim t^{-\gamma/2}\phi\left(\frac{x}{t^{(p+1)/2}}\right) \quad \text{as } t \to \infty, \tag{2}$$

where $\gamma = p + 1$ and the function on the right-hand side of (2) is a self-similar solution of equation (1) with $c(t) = t^p$ and $F \equiv 0$. Thus, a curious phenomenon, *universality*, is characterized: solutions of many different equations, but differing only in the nonlinear term $F(u)$, the higher-order asymptotics of $c(t)$, or both, nevertheless share the same asymptotic scaling behavior, given by a self-similar solution of the time-dependent diffusion equation $u_t = t^p\, u_{xx}$. Thus, it can be said that such equations belong to the same *universality class*, in that all of the members of this class exhibit the same asymptotic behavior, insofar as the scaling behavior of their solutions (with initial data in suitable classes) is the same.

"Subcritical" (or relevant) and "critical" (or marginal) nonlinearities, namely those varying as $F(u) \sim u^\alpha$ with $\alpha < (p+3)/(p+1)$ and $\alpha = (p+3)/(p+1)$, respectively, in the limit $u \to 0$, are analyzed numerically in [6]. This analysis does not assume $F(u)$ to be analytic at $u = 0$. In marked contrast with the supercritical case, in the subcritical case the asymptotic scaling behavior of solutions is strongly affected by the nonlinear term $F(u)$. In particular, the decay exponent $\gamma = 2/(\alpha - 1)$ is determined by the leading-order term (u^α) in $F(u)$, and the function on the right-hand side of (2) is now a self-similar solution of the time-dependent reaction-

diffusion equation $u_t = t^p\, u_{xx} - u^\alpha$. Thus, the phenomenon of universality is again visible. Equations differing only in the higher-order asymptotics of $F(u)$, $c(t)$, or both, fall in the same universality class.

The critical case is peculiar. It marks the crossover from a scaling regime controlled (mostly) by diffusion (supercritical case) to a scaling regime strongly influenced, and in certain aspects determined, by nonlinearity (subcritical case). Thus, in the critical case neither diffusion nor nonlinearity prevails, and the scaling regime which is observed bears some features of the supercritical one (same γ and scaling function ϕ), but acquires an extra logarithmic decay factor, the imprint of the critical nonlinearity.

The following heuristic arguments motivate the critical case, $\alpha = (p+3)/(p+1)$. Anticipating that solutions $u \to 0$ as $t \to \infty$, it is possible to simplify the analysis by taking $F(u) = \lambda u^\alpha$. This can be thought of as being an approximation near $u = 0$. With this choice, and recognizing that it is the large-time regime which we seek to understand, we simplify the equation to

$$u_t = t^p u_{xx} + \lambda u^\alpha.$$

Under the parabolic scaling $x \mapsto L^{(p+1)/2}x$, $t \mapsto Lt$, $u \mapsto L^{(p+1)/2}u$, $L \gg 1$, this equation becomes

$$u_t = t^p u_{xx} + L^{1+(1-\alpha)(p+1)/2}\lambda u^\alpha.$$

Thus, provided $\alpha > (p+3)/(p+1)$, as we iterate such scaling transformation we end up with an equation where the nonlinear term decreases by a factor $L^{1+(1-\alpha)(p+1)/2}$ at each rescaling. Consequently, as the number of rescalings $n \to \infty$ (equivalently, $t \to \infty$), the linear diffusion term dominates the nonlinear term and we may expect solutions to decay as $t^{-(p+1)/2}$, the rate determined by the linear diffusion. This argument fails when $\alpha = (p+3)/(p+1)$, which we dub the critical case.

We now state precisely the main result of this paper. For this purpose we introduce

the spaces

$$\mathcal{B}_q \equiv \{f : \mathbb{R} \to \mathbb{R} \mid \hat{f}(w) \in C^1(\mathbb{R}) \text{ and } \|f\| < \infty\}, \quad q > 1,$$

with norm $\|f\| = \sup_{w\in\mathbb{R}} \left[(1+|w|^q)\left(\left|\hat{f}(w)\right| + \left|\hat{f}'(w)\right|\right)\right]$ and

$$B^{(\infty)} \equiv \{u : \mathbb{R} \times [1, +\infty) \to \mathbb{R} \mid u(\cdot, t) \in \mathcal{B}_q \text{ for all } t \geq 1 \text{ and } \|u\|_\infty < \infty\},$$

where $\|u\|_\infty = \sup_{t\geq 1} \|u(\cdot, t)\|$.

Consider the following initial value problem (IVP)

$$\begin{cases} u_t = c(t)u_{xx} + \lambda F(u), & t > 1, \quad x \in \mathbb{R} \\ u(x, 1) = f(x), & x \in \mathbb{R} \end{cases} \tag{3}$$

and assumptions:

$(H1)$ $f \in \mathcal{B}_q$ for some $q > 1$;

$(H2)$ $c(t) > 0$ for $t > 1$ and $c(t) = t^p + o(t^p)$ as $t \to \infty$, with $p > 0$;

$(H3)$ $F(u) = \sum_{j\geq\alpha} a_j u^j$ analytic at $u = 0$, with $\alpha > (p + 3)/(p + 1)$.

We shall prove the following.

Theorem 1.1 *Assume* $(H1) - (H3)$. *Then there exists an* $\varepsilon > 0$ *such that, for* $\|f\| < \varepsilon$, *one can find* $B \subset B^{(\infty)}$ *such that the IVP (3) has a unique solution* $u \in B$ *which satisfies, for some constant* A,

$$\lim_{t\to\infty} \left\| \sqrt{t^{p+1}}u(\sqrt{t^{p+1}}\cdot, t) - Af_p^*(\cdot) \right\| = 0, \tag{4}$$

with $f_p^*(x) = \sqrt{\frac{p+1}{4\pi}}\ e^{-\frac{(p+1)}{4}x^2}$.

Our proof relies on the Renormalization Group (RG) approach. RG methods were originally introduced, and proved to be very useful, in quantum field theory [4, 15] and statistical mechanics [27, 28]. Their application to the asymptotic analysis of deterministic differential equations (both ODEs and PDEs) was initiated and

developed by Goldenfeld, Oono and collaborators [12, 18, 19]. See [17] and [25] for detailed accounts. The mathematical aspects of the method were rigorously established by Bricmont, Kupiainen and collaborators [8, 9]. See also [11, 23].

In the RG approach the long-time behavior of solutions to PDEs is related to the existence and stability of fixed points of an appropriate RG transformation. The definition of an RG transformation involves two basic steps. The first step is the integration (solution) of the PDE over a finite time interval; its purpose is to eliminate the "small time" information in the problem (coarse graining). The second step is rescaling, to change the time scales in proportion to those eliminated (by integration), so that a nominally constant time scale is under study. The iterative application of the RG transformation progressively evolves the solution in time and at the same time renormalizes the terms of the PDE. Once a proper RG transformation has been found for a particular problem, these terms are divided into two types: neutral and irrelevant, according to whether their magnitude is unchanged or decreases with each RG iteration. The irrelevant terms iterate to zero and the dynamics at large times is then controlled by the neutral terms. This accounts for the observed universal scaling behavior of solutions as they decay to zero. Thus, the RG provides a natural framework in which to understand universality.

Our results contribute to a large body of literature devoted to the study of long-time asymptotics of nonlinear PDEs. The work of Barenblatt and collaborators [3] has had a major impact in this field of study, specially in elucidating the importance, as well as intricacies, of self-similarity in intermediate asymptotics. Our analysis follows closely the one in [9]. See also [5, 10, 21]. In spirit, our contribution relates also to the work summarized in [1, 26].

Our analysis can easily be extended in a number of directions. These include equations in more than one space dimension, nonlinearities involving u as well

as derivatives of u (as in equation (1)), and nonlinearities with time-dependent coefficients. The modifications needed in each case are straightforward. For instance, if the nonlinearity is of the form $d(t)F(u)$, with $d(t) \sim t^r$ as $t \to \infty$ and $F(u) \sim u^\alpha$ as $u \to 0$, the elementary scaling argument presented above suggests that the critical exponent is now $\alpha = (p + 3 + 2r)/(p + 1)$. With this proviso Theorem 1.1 should also hold in this case.

The rest of this paper is organized as follows. In Section 2 we establish the existence and uniqueness of solutions for problem (3) over a finite time interval. In Section 3 we employ the RG approach to extend this result to an infinite time interval and obtain the long-time asymptotics of solutions, thereby proving Theorem 1.1.

2 Local Existence and Uniqueness

In this section we prove the existence and uniqueness of solutions for problem (3) over a finite time interval using a fixed-point argument. In the next section we employ the RG iterative procedure [9] to extend this local result over an infinite time interval. In the process we obtain upper bounds that lead to the limit (4).

As a preliminary remark, we note that generically the constants obtained in this and in the next sections depend on $q > 1$, the function $c(t)$ and the coefficients a_j of $F(\cdot)$. However, for simplicity of notation we omit this dependence. Also, without loss of generality we assume $\lambda \in [-1, 1]$ so that the estimates obtained will be valid uniformly with respect to λ.

We start with the definition of certain spaces and operators that will be used throughout this paper. For $L > 1$, we introduce the space

$$B^{(L)} \equiv \{u : \mathbb{R} \times [1, L] \to \mathbb{R} \mid u(\cdot, t) \in \mathcal{B}_q \text{ for all } t \in [1, L]\}, \tag{5}$$

with norm $\|u\|_L = \sup_{t\in[1,L]} \|u(\cdot,t)\|$. Next, let

$$u_f(x,t) = \frac{1}{\sqrt{4\pi s(t)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4s(t)}} f(y)\,\mathrm{d}y,$$

$$N(u)(x,t) = \lambda \int_1^t \frac{1}{\sqrt{4\pi[s(t)-s(\tau)]}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4[s(t)-s(\tau)]}} F(u(y,\tau))\,\mathrm{d}y\mathrm{d}\tau, \tag{6}$$

with $s(t) = \int_1^t c(v)\,\mathrm{d}v$, and define the operator

$$T(u) \equiv u_f + N(u). \tag{7}$$

We shall prove that the operator T has a unique fixed point. This is equivalent to proving the existence and uniqueness of solutions to IVP (3). Specifically, we prove existence and uniqueness of solutions in the ball B_f defined below, provided $\|f\|$ is sufficiently small. The introduction of B_f is a necessary step since we only stipulate the behavior of $F(u)$ near $u = 0$. So we define

$$B_f \equiv \left\{ u \in B^{(L)} : \|u - u_f\|_L \leq \|f\| \right\}. \tag{8}$$

Theorem 2.1 *Assume* $(H1)$-$(H3)$ *and let* $L > 1$. *Then there is an* $\varepsilon > 0$ *such that the IVP (3) has a unique solution* u *in* B_f, *if* $\|f\| < \varepsilon$.

The result follows immediately from Lemmas 2.1 and 2.2, since the former establishes that T maps B_f into itself and the latter that T is a contraction. We note that Theorem 2.1 also holds under the weaker assumption $\alpha \geq 2$, instead of $\alpha > (p+3)/(p+1)$ as stated in $(H3)$. However, the latter is necessary for the proof of Theorem 1.1 in the next section.

Lemma 2.1 *Let* $L > 1$. *There is an* $\varepsilon' > 0$ *such that* $\|N(u)\|_L < \|f\|$ *for all* $u \in B_f$, *if* $f \in \mathcal{B}_q$ *and* $\|f\| < \varepsilon'$.

Lemma 2.2 *For each* $L > 1$ *there is an* $\varepsilon'' > 0$ *such that* $\|N(u) - N(v)\|_L < \frac{1}{2}\|u - v\|_L$ *for all* $u, v \in B_f$, *if* $f \in \mathcal{B}_q$ *and* $\|f\| < \varepsilon''$.

Before proving Lemmas 2.1 and 2.2, we make some remarks and obtain estimates which are needed in the proof and also in Section 3.

Let us assume that the Taylor expansion of $F(u)$ at $u = 0$ has a finite radius of convergence ρ (the case $\rho = \infty$ is less restrictive). We argue that for $N(u)$ and $T(u)$ to be well defined it suffices to require that $u \in B_f$ with $f \in \mathcal{B}_q$ such that $\|f\| < [2C_q(1+\sqrt{s(L)})]^{-1}\rho$. Indeed, we have to check that $|u(x,t)| < \rho$ for all $x \in \mathbb{R}$ and $t \in [1, L]$ and this will follow by comparison of different norms. First, notice that $\mathcal{B}_q \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by elementary Fourier transform calculations. For instance, for all $x \in \mathbb{R}$

$$|h(x)| \leq \sup_x \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{h}(w) e^{iwx}| \, \mathrm{d}w \leq \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\|h\|}{1 + |w|^q} \, \mathrm{d}w = C_q \|h\|.$$

So, if $u \in B^{(L)}$, we have $u(\cdot, t) \in \mathcal{B}_q$ and

$$\sup_x |u(x,t)| \leq C_q \|u\|_L$$

for all $t \in [1, L]$. Now for $f \in \mathcal{B}_q$ and $u \in B_f$ we have

$$\|u\|_L \leq 2\left(1 + \sqrt{s(L)}\right)\|f\|. \tag{9}$$

Define $\varphi(\tau) \equiv s(t) - s(t-\tau)$ (recall $s(t) = \int_1^t c(v) \; \mathrm{d}v$) and $J(w,t) \equiv \int_0^{t-1} w\varphi(\tau)e^{-\varphi(\tau)w^2} \, \mathrm{d}\tau$ for $t \geq 1$ and $w \in \mathbb{R}$. The following estimate holds:

$$|J(w,t)| \leq (t-1)\sqrt{s(t)}. \tag{10}$$

Indeed, taking $x = \sqrt{\varphi(\tau)}w$ and using that $xe^{-x^2} \leq 1$ for all $x \in \mathbb{R}$,

$$|J(w,t)| = \left| \int_0^{t-1} \sqrt{\varphi(\tau)} x e^{-x^2} \, \mathrm{d}\tau \right| \leq \int_0^{t-1} \sqrt{\varphi(\tau)} \, \mathrm{d}\tau.$$

Since the integrand is a continuous function in $[0, t-1]$ and $\varphi(\tau)$ is an increasing function (notice that $c(t) \geq 0$), we can bound the last integral by $(t-1)\sqrt{\varphi(t-1)} = (t-1)\sqrt{s(t) - s(1)}$ and, since $s(1) = 0$, we have (10).

Also, if $q > 1$ and $w \in \mathbb{R}$,

$$I(w) \equiv \int_{\mathbb{R}} \frac{1}{1 + |x|^q} \cdot \frac{1}{1 + |x-w|^q} \, \mathrm{d}x \leq \frac{C}{1 + |w|^q}, \tag{11}$$

where

$$C = C(q) = (2^{q+1} + 3) \int_{\mathbb{R}} \frac{1}{1 + |x|^q} \, \mathrm{d}x. \tag{12}$$

Now, motivated by hypothesis $(H3)$, let C be given by (12) and define the sums

$$\begin{aligned} S_0(z) &\equiv \sum_{j \geq \alpha} \left(\frac{C}{2\pi}\right)^{j-1} |a_j| z^j, && (13) \\ S_1(z) &\equiv \sum_{j \geq \alpha} \left(\frac{C}{2\pi}\right)^{j-1} |a_j| z^{j-2}, && (14) \\ S_2(z) &\equiv \sum_{j \geq \alpha} \left(\frac{C}{2\pi}\right)^{j-1} j|a_j| z^{j-2}. && (15) \end{aligned}$$

Notice that the radius of convergence of these sums is $(2\pi\rho)/C < \rho$. We will now consider only those functions u such that $|u(x,t)| < \pi\rho/C \equiv \rho_0$ for all $x \in \mathbb{R}$ and $t \in [1, L]$. Invoking (9), it thus suffices to take f such that

$$\|f\| < \left[2C_q(1 + \sqrt{s(L)})\right]^{-1} \rho_0. \tag{16}$$

Proof of Lemma 2.1: Taking the Fourier transform of $N(u)$ yields

$$\widehat{N(u)}(w,t) = \lambda \sum_{j \geq \alpha} a_j \int_0^{t-1} e^{-\varphi(\tau)w^2} \widehat{u^j}(w, t - \tau) \, \mathrm{d}\tau.$$

Writing $\widehat{u^j}$ as convolutions of $\hat{u}$, each term in the sum above is of the form

$$\frac{a_j}{(2\pi)^{j-1}} \int_0^{t-1} \mathrm{d}\tau e^{-\varphi(\tau)w^2} \int_{\mathbb{R}^{j-1}} \hat{u}(w - p_1)\hat{u}(p_1 - p_2) \cdots \hat{u}(p_{j-1}) \, \mathrm{d}p_1 \cdots \mathrm{d}p_{j-1}, \tag{17}$$

where we have omitted the dependence of $\hat{u}$ on $t - \tau$. Since the absolute value of $\hat{u}$ is bounded by $\|u\|_L/(1 + |w|^q)$, (17) can be upper-bounded by

$$\frac{|a_j|}{(2\pi)^{j-1}} \|u\|_L^j \int_0^{t-1} \mathrm{d}\tau e^{-\varphi(\tau)w^2} \int_{\mathbb{R}^{j-1}} \frac{1}{1 + |w - p_1|^q} \cdots \frac{1}{1 + |p_{j-1}|^q} \, \mathrm{d}p_1 \cdots \mathrm{d}p_{j-1}.$$

Here the integrals over $\mathbb{R}$ no longer depend on τ and we can bound the exponential by one to obtain $t-1$ as an upper bound for the integral with respect to τ. Therefore, using (11),

$$\left|\widehat{N(u)}(w,t)\right| \leq |\lambda| \, \frac{t - 1}{1 + |w|^q} \, S_0(\|u\|_L), \tag{18}$$

where S_0 is the sum given by (13). Similarly, the derivative of (17) with respect to w can be bounded above by

$$(|2J(w,t)|+t-1)\,\frac{|a_j|}{(2\pi)^{j-1}}\|u\|_L^j\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\frac{1}{1+|w-p_1|^q}\cdots\frac{1}{1+|p_{j-1}|^q}\,\mathrm{d}p_1\cdots\mathrm{d}p_{j-1}$$

and using (10) and (11) we conclude that

$$|\widehat{N(u)}'(w,t)|\leq|\lambda|\frac{(2\sqrt{s(t)}+1)}{1+|w|^q}(t-1)S_0(\|u\|_L). \tag{19}$$

Estimates (18) and (19), together with the monotonicity of $s(t)$ and the inequality $\alpha\geq(p+3)/(p+1)$, then imply that

$$\|N(u)\|_L \quad\leq\quad 2|\lambda|(\sqrt{s(L)}+1)(L-1)\|u\|_L^2\sum_{j\geq\alpha}\left(\frac{C}{2\pi}\right)^{j-1}|a_j|\|u\|_L^{j-2}.$$

We can now bound the sum above by its value when $\|u\|_L=\rho_0$ and use (9) to obtain

$$\|N(u)\|_L\leq C'|\lambda|\ \|f\|^2, \tag{20}$$

where

$$C'=C'(L,q,F,c)=8\left(\sqrt{s(L)}+1\right)^3(L-1)S_1(\rho_0). \tag{21}$$

Finally, recalling that $|\lambda|\leq1$, invoking (16), and defining

$$\varepsilon'\equiv\min\left\{{C'}^{-1},[2C_q(1+\sqrt{s(L)})]^{-1}\rho_0\right\},$$

where C' is given by equation (21), we conclude that $\|N(u)\|_L<\|f\|$ whenever $\|f\|<\varepsilon'$. ∎

Proof of Lemma 2.2: Consider functions u and v such that $\|u\|_L<\rho_0$ and $\|v\|_L<\rho_0$. Then,

$$[\widehat{N(u)}-\widehat{N(v)}](w,t)=\lambda\sum_{j\geq\alpha}a_j\int_0^{t-1}\mathrm{d}\tau e^{-\varphi(\tau)w^2}[\widehat{u^j}-\widehat{v^j}](w,t-\tau)\equiv\lambda\sum_{j\geq\alpha}D_j,$$

where D_j can be written as

$$D_j=\frac{a_j}{(2\pi)^{j-1}}\int_0^{t-1}\mathrm{d}\tau e^{-\varphi(\tau)w^2}[(\hat{u}*\cdots*\hat{u})-(\hat{v}*\cdots*\hat{v})](w,\tau).$$

Here there are $j-1$ convolutions of $\hat{u}$ and $j-1$ of $\hat{v}$. We add and subtract in the integrand the term $\hat{v} * \hat{u} * \cdots * \hat{u}$, with $j-2$ convolutions of $\hat{u}$, to get

$$
\begin{aligned}
D_j &= \frac{a_j}{(2\pi)^{j-1}} \int_0^{t-1} e^{-\varphi(\tau) w^2} [(\hat{u}-\hat{v}) * \hat{u} * \cdots * \hat{u}](w,\tau)\, \mathrm{d}\tau + \\
&+ \frac{a_j}{(2\pi)^{j-1}} \int_0^{t-1} e^{-\varphi(\tau) w^2} [\hat{v} * (\hat{u} * \cdots * \hat{u} - \hat{v} * \cdots * \hat{v})](w,\tau)\, \mathrm{d}\tau.
\end{aligned}
$$

The first integral can be bounded, in a manner similar to what was done in the proof of Lemma 2.1, by $(t-1)(1+|w|^q)^{-1}(2\pi)^{1-j} C^{j-1} |a_j| \|u\|_L^{j-1} \|u-v\|_L$. To estimate the second integral we rewrite it, after adding and subtracting appropriate terms, as a sum of two integrals, one of which can be bounded as above and the other can be split into two other integrals. This procedure ends after $j-1$ steps, when we obtain

$$
D_j \leq \frac{(t-1)}{1+|w|^q} \cdot \frac{C^{j-1}|a_j|}{(2\pi)^{j-1}} \|u-v\|_L \left(\|u\|_L^{j-1} + \|v\|_L \|u\|_L^{j-2} + \cdots + \|v\|_L^{j-1} \right).
$$

Note that since the norms of u and v in $B^{(L)}$ are less than ρ_0, the sum over $j \geq \alpha$ of the right-hand side of the inequality above is convergent. In addition, we can factor $\|u\|_L$ or $\|v\|_L$ and the remaining sum will still be convergent. Similarly, each term of the derivative with respect to w of the difference $N(u) - N(v)$ can be written as a sum of two integrals, which we bound using the same procedure as before. Therefore,

$$
\|N(u) - N(v)\|_L \quad \leq \quad C'' |\lambda| \|f\| \|u-v\|_L,
$$

where

$$
C'' = C''(L,q,F,c) = 4(\sqrt{s(L)}+1)^2 (L-1) S_2(\rho_0). \tag{22}
$$

Since $|\lambda| \leq 1$, defining

$$
\varepsilon'' \equiv \min \left\{ (2C'')^{-1}, [2C_q(1+\sqrt{s(L)})]^{-1} \rho_0 \right\},
$$

where C'' is given by (22), the lemma is proved if we take $\|f\| < \varepsilon''$. ∎

We note for future use that, since $S_2(\rho_0) > S_1(\rho_0)$, we may take

$$
C_0 \equiv 8(\sqrt{s(L)}+1)^3 (L-1) S_2(\rho_0), \tag{23}
$$

and it is enough to consider ε in Theorem 2.1 defined by

$$\varepsilon \equiv \min\left\{(2C_0)^{-1}, \left[2C_q\left(\sqrt{s(L)}+1\right)\right]^{-1}\rho_0\right\}. \tag{24}$$

3 Global Existence, Uniqueness and Asymptotic Behavior

It follows from Theorem 2.1 that, given $L > 1$, there exists an $\varepsilon > 0$ such that the IVP (3) has a unique solution u in B_f for any $f \in \mathcal{B}_q$ with $\|f\| < \varepsilon$. Therefore,

$$f_1(x) \equiv L^{\frac{(p+1)}{2}} u\left(L^{\frac{(p+1)}{2}} x, L\right) \tag{25}$$

is a well defined element of $\mathcal{B}_q$. The right-hand side of (25) defines an operator, $R_{L,0}$, acting on the ball $\{f \in \mathcal{B}_q : \|f\| < \varepsilon\}$, which maps the initial condition f to f_1. We dub $R_{L,0}$ the *Renormalization Group operator* associated to problem (3).

The RG operator just defined was introduced in [9]; its iteration comprises the RG method for the asymptotic analysis of solutions. The basic idea of this method is to reduce the long-time-asymptotics problem to the analysis of a sequence of finite-time problems obtained by iterating the RG operator. In more detail, first consider problem (3) and, as in Section 2, restrict the initial data so that this problem has a unique solution. Then, apply $R_{L,0}$ to the initial data f to produce f_1, the initial data for a new, *renormalized* IVP. It is expected that this procedure can be iterated ad infinitum to generate a sequence of finite-time IVPs, whose initial conditions f_n are obtained by iterating the RG operator n times.

We now outline the RG method for the nonlinear problem (3). See also [6, 7]. Assume that the solution u to IVP (3) is globally well defined and let $L > 1$ be fixed. We consider a sequence $\{u_n\}_{n=0}^{\infty}$ of rescaled functions defined by

$$u_n(x,t) \equiv L^{n(p+1)/2} u\left(L^{n(p+1)/2} x, L^n t\right), \tag{26}$$

with $t \in [1, L]$. A direct calculation reveals that u_n satisfies the renormalized IVP:

$$\begin{cases} \partial_t u_n = c_n(t)\partial_x^2 u_n + \lambda_n F_n(u_n), & t \in [1,L],\ x \in \mathbb{R}, \\ u_n(x,1) = f_n(x), & x \in \mathbb{R}, \end{cases} \tag{27}$$

where $c_n(t) = L^{-np}c(L^n t)$, $\lambda_n = L^{n[p+3-\alpha(p+1)]/2}\lambda$,

$$F_n(v) = \sum_{j\geq\alpha}\left[L^{n(\alpha-j)(p+1)/2}a_j\right]v^j$$

and

$$f_n(x) = u_n(x,1) = L^{n(p+1)/2}u\left(L^{n(p+1)/2}x, L^n\right). \tag{28}$$

Comparing (28) and (4), it becomes clear that proving the asymptotic limit may be reduced to proving the convergence of $\{f_n\}$, which motivates the definition of the RG operator. Let $g \in \mathcal{B}_q$ and for a given $n \geq 0$ assume that the IVP (27) with initial condition g has a unique solution u_n. Then, rescale $u_n(\cdot, L)$ to obtain

$$(R_{L,n}g)(x) \equiv L^{(p+1)/2}u_n\left(L^{(p+1)/2}x, L\right), \tag{29}$$

which defines the RG operator. The index n in the above definition is justified since the operator depends on the evolution equation considered. Now if we consider IVP (27) with initial data f_n given by (28), then it is an immediate consequence of these definitions that the sequence $\{f_n\}$ satisfies

$$f_0 = u(\cdot, 1) \qquad \text{and} \qquad f_{n+1} = R_{L,n}f_n. \tag{30}$$

Our goal from now on is to make the above heuristic argument rigorous. We shall prove that under hypotheses $(H1)-(H3)$, if the initial data is sufficiently small, problem (27) has a unique solution for each n so that the iterative RG method can be applied to furnish the asymptotic behavior of the solution to IVP (3).

In Lemma 3.1 we proceed as in the proof of Theorem 2.1 to obtain local existence and uniqueness of solutions for each problem (27). To state the lemma, consider the space $B^{(L)}$ defined by (5) and, if f_n is the initial data of problem (27), define the space $B_{f_n} = \{u_n \in B^{(L)} : \|u_n - u_{f_n}\| \leq \|f_n\|\}$ and the operator $T_n(u_n) \equiv u_{f_n} + N_n(u_n)$, where u_{f_n} is the solution of (27) with $\lambda_n = 0$ and

$$N_n(u_n)(x,t) = \lambda_n \int_0^{t-1}\int \frac{e^{-\frac{(x-y)^2}{4[s_n(t)-s_n(t-\tau)]}}}{\sqrt{4\pi[s_n(t)-s_n(t-\tau)]}}F_n(u_n(y,t-\tau))\,\mathrm{d}y\mathrm{d}\tau, \tag{31}$$

where

$$s_n(t) = \int_1^t c_n(v)\,\mathrm{d}v = \frac{t^{p+1}-1}{p+1} + r_n(t). \tag{32}$$

Define also the constant C_n by

$$C_n \equiv 8(\sqrt{s_n(L)}+1)^3(L-1)S_2(\rho_0), \tag{33}$$

where $S_2(\rho_0)$ is given by (15), with $z = \rho_0$.

Lemma 3.1 *Given $n \in \mathbb{N}$ and $L > 1$, there exists an $\varepsilon_n > 0$ such that if $\|f_n\| < \varepsilon_n$, then the IVP (27) has a unique solution $u_n(x,t)$ in B_{f_n}. Furthermore, f_{n+1} given by (30) is a well defined element of the $\mathcal{B}_q$ space.*

Proof: (Notice that if $n = 0$ and $f_0 \equiv f$, Lemma 3.1 is just Theorem 2.1.) We must prove that the operator T_n is a contraction in B_{f_n}, therefore obtaining a unique solution u_n in B_{f_n}. First, following closely the arguments in the proof of Lemma 2.1, the constraint $L > 1$ and the definitions of F_n and $s_n(t)$ imply that

$$\|N_n(u_n)\|_L \le C_n L^{n[p+3-\alpha(p+1)]/2}\|f_n\|^2 \tag{34}$$

and that

$$\|N_n(u_n) - N_n(v_n)\| \le C_n L^{n[p+3-\alpha(p+1)]/2}\|f_n\|\|u_n - v_n\|,$$

where C_n is given by (33). The condition for u_n to be in the region of analyticity of F_n is now that $\|f_n\| < [2C_q(1+\sqrt{s_n(L)})]^{-1}\rho_0$. Since $p+3-\alpha(p+1) < 0$, defining

$$\varepsilon_n \equiv \min\left\{(2C_n)^{-1}, [2C_q(1+\sqrt{s_n(L)})]^{-1}\rho_0\right\}, \tag{35}$$

if $\|f_n\| < \varepsilon_n$, we obtain $\|N_n(u_n)\|_L < \|f_n\|$ and $\|N_n(u_n) - N_n(v_n)\| < \frac{1}{2}\|u_n - v_n\|_L$ for all $u_n, v_n \in \mathcal{B}_{f_n}$. This proves that the IVP (27) has a unique solution $u_n(x,t)$ in B_{f_n} and, therefore, $f_{n+1} \equiv L^{(p+1)/2}u_n\left(L^{(p+1)/2}x, L\right)$ is well defined. ∎

We have proved that if $\|f_n\| < \varepsilon_n$, then $R_{L,n}f_n$ is well defined. To simplify the notation, let $\nu_n(x) \equiv N_n(u_n)(x,L)$ (cf. (31), where $N_0(u) = N(u)$). Then, the

solution to the IVP (27) at time $t = L$ can be written as $u_n(x,L) = u_{f_n}(x,L) + \nu_n(x)$ and we have

$$(R_{L,n} f_n)(x) = R_L^0 f_n(x) + L^{(p+1)/2} \nu_n(L^{(p+1)/2} x), \tag{36}$$

where $R_L^0 \equiv R_{L,0}^0$ and $(R_{L,n}^0 f_n)(x) \equiv L^{(p+1)/2} u_{f_n}\left(L^{(p+1)/2} x, L\right)$. We see that (36) splits the RG operator into two parts, which we dub the linear and the nonlinear parts. Our analysis focus first on the linear part; the nonlinear part is driven to zero under hypothesis $(H3)$ and, thereby, does not contribute to the asymptotic regime, as we shall prove.

It follows from the definition of $R_{L,n}^0$ and from the integral representation of u_{f_n} that if $g \in \mathcal{B}_q$ is the initial data of IVP (27) with $\lambda_n = 0$, then the Fourier Transform of $R_{L,n}^0 g$ is given by

$$\mathcal{F}(R_{L,n}^0 g)(w) = \widehat{g}(L^{-(p+1)/2} w) e^{-w^2 s_n(L)/L^{p+1}}, \tag{37}$$

where s_n is defined in (32). Applying equation (37) inductively and using that $s_0(L^n) + L^{n(p+1)} s_n(L) = s_0(L^{n+1})$ for all $n = 1, 2, \ldots$, it is easy to prove that the linear RG operator has the semi-group property. Also, if $g = f_p^*$, it follows from equation (37) and definition (32) with $n = 0$ that

$$\mathcal{F}(R_L^0 f_p^*)(w) = e^{-w^2/(p+1)} e^{-w^2 r(L)/L^{p+1}}, \tag{38}$$

where $r(L) \equiv r_0(L)$ (see (32) for the definition of $r_0(L)$) and we have used that $\hat{f}_p^*(w) = e^{-w^2/(p+1)}$. Taking the inverse Fourier Transform on both sides of equation (38) we conclude that $f_p^*(x)$ is a fixed point of the RG operator if and only if $r(L) \equiv 0$, which is valid if we take $c(t) = t^p$. The next lemma shows that if $r(t) \not\equiv 0$, f_p^* is still the long-time asymptotic limit of $R_L^0 f_p^*$ and, therefore, it will be an *asymptotic fixed point* of the linear RG operator.

Lemma 3.2 *There is a constant $M = M(p,q)$ and an $n_0 \in \mathbb{N}$ such that*

$$\|R_{L^n}^0 f_p^* - f_p^*\| \leq M |L^{-n(p+1)} r(L^n)| \tag{39}$$

for all $n > n_0$. In particular, $\|R^0_{L^n} f^*_p - f^*_p\| \longrightarrow 0$ as $n \to \infty$.

Proof: Since $r(t) = o(t^{p+1})$, from the semi-group property of R^0_L and from equation (38), with L replaced by L^n, we have pointwise convergence in Fourier space. To prove the theorem, we have to show convergence in $\mathcal{B}_q$. This will be done at the same time that we estimate the rate of convergence. From (38),

$$\left|[\mathcal{F}(R^0_{L^n} f^*_p - f^*_p)](w)\right| \leq w^2 \left|\frac{r(L^n)}{L^{n(p+1)}}\right| e^{-w^2\left[\frac{1}{p+1} - \left|\frac{r(L^n)}{L^{n(p+1)}}\right|\right]}$$

and

$$\left|[\mathcal{F}(R^0_{L^n} f^*_p - f^*_p)]'(w)\right| \leq 2\left[\frac{|w|^3}{p+1} + |w|\right] \left|\frac{r(L^n)}{L^{n(p+1)}}\right| e^{-w^2\left[\frac{1}{p+1} - \left|\frac{r(L^n)}{L^{n(p+1)}}\right|\right]}.$$

Since $r(t) = o(t^{p+1})$, there exists an $n_0 > 0$ such that $\left|r(L^n)L^{-n(p+1)}\right| < [2(p+1)]^{-1}$ for all $n > n_0$. Multiplying the inequalities above by $(1 + |w|^q)$ and defining $M \equiv \max_w (1+|w|^q) e^{-[2(p+1)]^{-1} w^2} [2|w| + w^2 + 2|w|^3/(p+1)]$ we obtain (39). Letting $n \to \infty$ finishes the proof. ∎

In the next lemma we prove that if L is sufficiently large, then the linear RG operator $R^0_{L,n}$ is a contraction in the space of functions $g \in \mathcal{B}_q$ such that $\hat{g}(0) = 0$. This result will be used to obtain the estimates of the Renormalization Lemma 3.4, which will allow us to prove Theorem 1.1.

Lemma 3.3 (Contraction Lemma) *There exist constants $L_1 > 1$ and $C > 0$ such that the inequality*

$$\left\|R^0_{L,n} g\right\| \leq L^{-(p+1)/2} C \|g\| \quad (n = 0, 1, \ldots) \tag{40}$$

holds if $L > L_1$ and $g \in \mathcal{B}_q$ satisfies $\hat{g}(0) = 0$.

Proof: We bound (37) and its derivative by

$$e^{-w^2 \frac{s_n(L)}{L^{p+1}}} \left[\left|\hat{g}\left(\frac{w}{L^{\frac{p+1}{2}}}\right)\right| + 2|w| \left|\frac{s_n(L)}{L^{p+1}}\right| \left|\hat{g}\left(\frac{w}{L^{\frac{p+1}{2}}}\right)\right| + L^{-(p+1)/2} \left|\hat{g}'\left(\frac{w}{L^{\frac{p+1}{2}}}\right)\right|\right].$$

From (32), it is easy to see that $r_n(t) = L^{-n(p+1)}[r(L^n t) - r(L^n)]$. Now, since $r(t) = o(t^{p+1})$, we conclude that there exists an $L_0 > 1$ such that, if $L > L_0$, then

$|L^{-(p+1)}r_n(L)| < [2(p+1)]^{-1}$ for all n. This together with the definition of s_n furnishes

$$\frac{1}{6(p+1)} \leq \frac{s_n(L)}{L^{p+1}} \leq \frac{3}{2(p+1)} \quad (n = 0, 1, 2, \ldots) \tag{41}$$

if $L > L_1 \equiv \max\{L_0, \sqrt[p+1]{3}\}$. Since $g \in \mathcal{B}_q$, from the definition of the $\mathcal{B}_q$ norm, $\left|\hat{g}'\left(L^{-(p+1)/2}w\right)\right| \leq \|g\|$ and since $\hat{g}(0) = 0$ it follows that $\left|\hat{g}\left(L^{-(p+1)/2}w\right)\right| \leq L^{-(p+1)/2}|w|\|g\|$. Using these bounds, if $L > L_1$ we obtain, for all $n = 0, 1, 2, \ldots,$

$$|(\widehat{R^0_{L,n}g})(w)| + |(\widehat{R^0_{L,n}g})'(w)| \leq \left(1 + |w| + 3(p+1)^{-1}|w|^2\right) e^{\frac{-w^2}{6(p+1)}} L^{-(p+1)/2}\|g\|.$$

Defining $C \equiv \sup_w \left(1 + |w| + 3(p+1)^{-1}|w|^2\right)(1 + |w|^q)e^{\frac{-w^2}{6(p+1)}}$ finishes the proof. ∎

The RG analysis involves decomposing the initial condition into two factors: one in the direction of the asymptotic fixed point of the RG operator and the other in a direction which is *irrelevant in the RG sense*. That is, when the RG operator is applied to the initial data, the irrelevant component is contracted for large L.

In Lemma 3.4 we decompose f_n given by (30) and obtain estimates needed later to prove Theorem 1.1. We will first assume that, given $k \in \mathbb{N}$, f_n is well defined for all $n = 0, 1, \ldots, k$, which is guaranteed if $\|f_n\| < \varepsilon_n$ for all $n = 0, 1, \ldots, k-1$ (cf. Lemma 3.1). Later, in Lemma 3.6 we prove that if f_0 is small enough, then the sequence $\{f_n\}$ given by (30) is well defined. Furthermore, notice that from the definition of the $\mathcal{B}_q$ norm and some estimates used in the proof of Lemma 3.3, we obtain constants $C_{p,q}$ and $K_{p,q}$ such that $\|f_p^*\| \leq C_{p,q}$ and, if $L > L_1$,

$$\|R^0_{L^n} f_p^*\| \leq K_{p,q}, \quad \forall n = 1, 2, \ldots \tag{42}$$

For the next lemmas, we will always refer to the constants $C_{p,q}$ and $K_{p,q}$ and to C and L_1 given in Lemma 3.3.

Lemma 3.4 (Renormalization Lemma) *Given $L > L_1$, suppose f_n is defined for $n = 0, 1, \ldots, k+1$ as specified in (30). We can then write*

$$f_0 = A_0 f_p^* + g_0, \quad f_{n+1} = A_{n+1} R^0_{L^{n+1}} f_p^* + g_{n+1} \quad (n = 0, 1, \ldots, k) \tag{43}$$

in terms of functions $g_n \in \mathcal{B}_q$, $\hat{g}_n(0) = 0$ and constants A_n, K which satisfy

$$\|g_{n+1}\| \leq CL^{-(p+1)/2}\|g_n\| + KL^{n[p+3-\alpha(p+1)]/2}\|f_n\|^2 \tag{44}$$

and

$$|A_{n+1} - A_n| \leq C_n L^{n[p+3-\alpha(p+1)]/2}\|f_n\|^2, \tag{45}$$

with C_n given by (33).

Proof: We first prove (43) inductively. Define g_0 by $f_0 = A_0 f_p^* + g_0$, with $A_0 = \hat{f}_0(0)$ and since $\widehat{f_p^*}(0) = 1$, we have $\hat{g}_0(0) = 0$. By hypothesis, f_1 is well defined by $R_{L,0}f_0$ and using representation (36) and the decomposition above for f_0 we can write $f_1 = A_1 R_L^0 f_p^* + g_1$, where $A_1 = A_0 + \hat{\nu}_0(0)$ and $g_1(x) = R_L^0 g_0(x) + L^{(p+1)/2}\nu_0(L^{(p+1)/2}x) - \hat{\nu}_0(0)R_L^0 f_p^*(x)$. It follows from the definition of R_L^0 that $\mathcal{F}(R_L^0 g_0)(0) = 0$ and $\mathcal{F}(R_L^0 f_p^*)(0) = 1$ and, therefore, $\hat{g}_1(0) = 0$, which proves (43) for $n = 0$. Now suppose (43) holds for $n = 0, \ldots, j-1$, where $j \leq k$. We will prove that it holds also for $n = j$. Using (43) with $n = j-1$, representation (36) and the semi-group property of the linear RG operator we obtain

$$f_{j+1}(x) = A_j R_{L^{j+1}}^0 f_p^*(x) + R_L^0 g_j(x) + L^{(p+1)/2}\nu_j(L^{(p+1)/2}x). \tag{46}$$

Defining

$$A_{j+1} \equiv A_j + \hat{\nu}_j(0) \tag{47}$$

and

$$g_{j+1}(x) \equiv R_L^0 g_j(x) + L^{(p+1)/2}\nu_j(L^{(p+1)/2}x) - \hat{\nu}_j(0)R_{L^{j+1}}^0 f_p^*(x), \tag{48}$$

we can write (46) as $f_{j+1} = A_{j+1}R_{L^{j+1}}^0 f_p^* + g_{j+1}$. From the induction hypothesis, $\hat{g}_j(0) = 0$ and therefore, from definition (48), since the Fourier Transforms of $R_L^0 g_j$ and $R_{L^{j+1}}^0 f_p^*$ at the origin are equal, respectively to $\hat{g}_j(0)$ and $\hat{f}_p^*(0)$, we obtain $\hat{g}_{j+1}(0) = 0$, which proves (43) for $n = 0, 1, \ldots, k$.

Recalling that $\nu_n(x) \equiv N_n(u)(x, L)$ and since estimate (34) holds for all n, using definition (47) we obtain (45) for $n = 0, 1, \ldots, k$. After a calculation similar

to the one in the proof of Lemma 2.1, we obtain $\|L^{(p+1)/2}\nu_n(L^{(p+1)/2}\cdot)\| \leq L^{(p+1)q/2}C_n L^{n[p+3-\alpha(p+1)]/2}\|f_n\|^2$ and using (42), $\|L^{(p+1)/2}\nu_n(L^{(p+1)/2}\cdot) - \hat{\nu_n}(0)R^0_{L^{n+1}}f^*_p\| \leq (L^{(p+1)q/2} + K_{p,q})C_n L^{n[p+3-\alpha(p+1)]/2}\|f_n\|^2$. Since $\hat{g}_n(0) = 0$ and $L > L_1$, from definition (48) and Lemma 3.3, we obtain

$$\|g_{n+1}\| \leq CL^{-(p+1)/2}\|g_n\| + \left(L^{(p+1)q/2} + K_{p,q}\right) C_n L^{n[p+3-\alpha(p+1)]/2}\|f_n\|^2 \quad (49)$$

for all $n = 0, 1, \ldots, k$. Now it follows from (41) that the constants C_n are uniformly bounded. In fact, defining

$$K \equiv 8(L-1)\left(\sqrt{\frac{3L^{p+1}}{2(p+1)}} + 1\right)^3 \left(L^{(p+1)q/2} + K_{p,q}\right) S_2(\rho_0), \quad (50)$$

then $C_n \leq K$ for all n and therefore we obtain inequality (44), which ends the proof.

∎

The estimates obtained in Lemma 3.4 are used to prove Theorem 1.1 in the following way: (45) guarantees that the sequence (A_n) is convergent and (44) is used to prove that the component g_n gets smaller as we increase n. This is so because of our definition of α or, in other words, because the nonlinear perturbation F of problem (3) is irrelevant. Before we apply Lemma 3.4, we have to prove that the initial data of each problem (27) is small enough and to do that we will define a recursive sequence (G_n) such that, for all n, $\|f_n\| \leq G_n\|f_0\|$. In Lemma 3.5 we prove that, under a certain condition, this sequence is bounded. Given $\delta \in (0, 1)$, let

$$L_\delta \equiv \max\{L_1, [2C(1 + C_{p,q})]^{2/(p+\delta)}\} \quad (51)$$

and for $L > L_\delta$, define

$$G \equiv 1 + K_{p,q}\sum_{j=0}^{\infty} L^{j(\delta-1)/2} < \infty, \quad (52)$$

$G_1 \equiv L^{(\delta-1)/2} + K_{p,q}(1 + C_0\|f\|)$ and G_{n+1}, for $n = 1, 2, 3, \ldots$, by the relation:

$$G_{n+1} \equiv L^{(\delta-1)(n+1)/2} + K_{p,q}\left(1 + C_0\|f\| + \sum_{j=1}^{n} C_j G_j^2 L^{j[p+3-\alpha(p+1)]/2}\|f\|\right),$$

where each C_j, with $j = 0, 1, 2, \ldots$, is given by equation (33), with $n = j$.

Lemma 3.5 *Let $\delta \in (0,1)$ be such that $\delta - 1 > p + 3 - \alpha(p+1)$ and let $L > L_\delta$, where L_δ is given by (51). Also, let K and G be given by (50) and (52), respectively, and suppose that f satisfies*

$$KG^2\|f\| < \frac{1}{2L^{(1-\delta)/2}}. \tag{53}$$

Then $G_{n+1} < G$ for all $n = 0, 1, 2, \dots$.

Proof: Since $L > 1$ and $G > 1$, it is straightforward from the fact that $C_0 \leq K$ and from condition (53) that $G_1 < G$. Now, suppose $G_{n+1} < G$, $\forall n = 1, 2, \dots, k-1$. From the definition of G_{n+1}, using the induction hypothesis and since $C_n \leq K$, $\forall n$,

$$G_{k+1} \leq L^{(\delta-1)(k+1)/2} + K_{p,q}\left(1 + K\|f\| + KG^2\|f\|\sum_{j=1}^{k} L^{j[p+3-\alpha(p+1)]/2}\right).$$

Now, from (53) and since $L > 1$ and $\delta - 1 > p + 3 - \alpha(p+1)$, we obtain $G_{k+1} \leq 1 + K_{p,q}\left(1 + L^{(\delta-1)/2} + \dots + L^{(k+1)(\delta-1)/2}\right) < G$, which completes the proof. ∎

In Lemma 3.6 we will obtain estimates for the rescaled solutions to IVP (27). In fact, we will define $\overline{\varepsilon} > 0$ such that, if the initial data f of problem (3) is in the ball of radius $\overline{\varepsilon}$, then there is a unique global solution to IVP (3). Furthermore, we will prove that, under certain hypotheses, the component g_n of the initial data f_n goes to zero when $n \to \infty$. This fact will be used to obtain the asymptotic behavior in Theorem 3.1. Before stating the lemma, we notice that from (41), if ε_n is given by (35) and

$$\sigma \equiv \min\left\{(2K)^{-1}, \left[2C_q\left(1 + \sqrt{\frac{3L^{p+1}}{2(p+1)}}\right)\right]^{-1}\rho_0\right\}, \tag{54}$$

then $\sigma < \varepsilon_n$ for all n. In the next Lemma we will refer to K, L_δ, G and σ given, respectively, by (50), (51), (52) and (54).

Lemma 3.6 *Let $L > L_\delta$ and $\delta \in (0,1)$ such that $\delta - 1 > p + 3 - \alpha(p+1)$. Then, there is $\overline{\varepsilon} > 0$ such that, if $\|f_0\| < \overline{\varepsilon}$, f_n given by (30) is well defined for all $n \geq 1$,*

$$\|f_n\| \leq G_n\|f_0\| \tag{55}$$

and if g_n is given by the decomposition (43), then,

$$\|g_n\| \leq \frac{\|f_0\|}{L^{n(1-\delta)/2}}. \tag{56}$$

Proof: The proof is by induction in n. First, define

$$\overline{\varepsilon} \equiv \min\left\{\sigma G^{-1}, [2KG^2L^{(1-\delta)/2}]^{-1}\right\}. \tag{57}$$

Since $G > 1$, we have $\|f_0\| < \sigma < \varepsilon_0$ and, from Lemmas 3.1 and 3.4, f_1 is well defined by $f_1 = A_1 R_L^0 f_p^* + g_1$ and g_1 satisfies (44) with $k = 0$. Therefore, since $f_0 = A_0 f_p^* + g_0$, we obtain $\|g_1\| \leq [C(1 + C_{p,q})L^{-(p+1)/2} + K\|f_0\|]\|f_0\|$. Since $L > L_\delta$, then, $2C(1 + C_{p,q})L^{-(p+1)/2} < L^{(\delta-1)/2}$ and since $G > 1$ and $\|f_0\| < \overline{\varepsilon}$, then $2K\|f_0\| < L^{(\delta-1)/2}$. Therefore, $\|g_1\| \leq L^{(\delta-1)/2}\|f_0\|$. Now, using decomposition (43) with $k = 0$ and the bound (42),

$$\|f_1\| \leq \left[(1 + C_0\|f_0\|)K_{p,q} + L^{(\delta-1)/2}\right]\|f_0\| = G_1\|f_0\|,$$

which proves the Theorem for $n = 1$. Now suppose there exists $k > 1$ such that (55) and (56) hold for all $n = 1, 2, \ldots, k$. We will prove that these estimates hold also for $n = k + 1$. From the induction hypothesis and Lemma 3.5, since $\|f_0\| < \overline{\varepsilon}$, we have $\|f_n\| \leq G\|f_0\| < \varepsilon_n$, $\forall n = 1, 2, \ldots, k$. Therefore, we can apply Lemma 3.4 to obtain estimate (44) with $n = k$. Then, using (55) and (56) with $n = k$, we get:

$$\|g_{k+1}\| \leq L^{(\delta-1)(k+1)/2}\left[\frac{C}{L^{(p+\delta)/2}} + \frac{L^{k[p+3-\alpha(p+1)]/2}}{L^{(\delta-1)(k+1)/2}}KG_k^2\|f_0\|\right]\|f_0\|.$$

Since $C_{p,q} > 0$ and $L > L_\delta$, then $CL^{-(p+\delta)/2} < 1/2$ and since $\|f_0\| < \overline{\varepsilon}$, using Lemma 3.5 we obtain (56) with $n = k+1$. By Lemma 3.4, f_{k+1} is well defined and can be represented by (43). Therefore, using the triangle inequality and (42),

$$\|f_{k+1}\| \quad \leq \quad \frac{\|f_0\|}{L^{(1-\delta)(k+1)/2}} + K_{p,q}\left(|A_0| + \sum_{j=0}^{k}|A_{j+1} - A_j|\right).$$

Now, since $|A_0| \leq \|f_0\|$ and $C_n \leq K$, for all n, applying estimates (45) and (55) for $n = 0, 1, 2, \ldots, k$ and using Lemma 3.5, we obtain (55) with $n = k+1$. In particular, $\|f_{k+1}\| < G\|f_0\| < \varepsilon_{k+1}$, which ends the proof. ∎

We have proved that for $\|f_0\| < \overline{\varepsilon}$ each IVP (27) has a unique solution u_n in B_{f_n}. To finish the proof we only need to concatenate these solutions to obtain a unique global solution to IVP (3).

We first extend the definition (5) of the $B^{(L)}$ space by considering the space

$$B^{(L^{n+1})} \equiv \left\{u : \mathbb{R} \times [L^n, L^{n+1}] \to \mathbb{R} \;\middle|\; \|u\|_{L^{n+1}} < \infty\right\}$$

with the norm $\|u\|_{L^{n+1}} = \sup_{t\in[L^n,L^{n+1}]} \|u(\cdot,t)\|$.

Now define $\{h_n\}$ by

$$h_0 \equiv f \qquad \text{and} \qquad h_{n+1} \equiv L^{-n(p+1)/2} u_n \left(L^{-n(p+1)/2}x, L\right)$$

and let u_{h_n} be the solution to IVP (27) with $\lambda_n = 0$ and initial condition h_n. Finally, define

$$B \equiv \left\{u \in B^{(\infty)} : \|u - u_{h_n}\|_{L^{n+1}} \leq \|h_n\|, \forall n\right\}, \tag{58}$$

where (abusing notation) $\|\cdot\|_{L^{n+1}}$ denotes the seminorm induced by the obvious projection from $B^{(\infty)}$ onto $B^{(L^{n+1})}$.

Corollary 3.1 *Under the hypotheses of Lemma 3.6 the IVP (3) has a unique solution $u \in B$. In this case, the RG transformation has the* "semi-group property"*:* $R_{L^n,0}f = R_{L,n-1} \circ \cdots \circ R_{L,1} \circ R_{L,0}f$ *for all* $n \geq 1$.

Proof: From Lemma 3.6, since $\|f_0\| < \overline{\varepsilon}$, then $\|f_n\| < \varepsilon_n$ for all n and using Lemma 3.1, we obtain the existence and uniqueness of solutions to problems (27) in B_{f_n}. Now define $u(x,t) \equiv L^{-n(p+1)/2}u_n\left(L^{-n(p+1)/2}x, L^{-n}t\right), \quad \forall t \in [L^n, L^{n+1}]$ and take $y = L^{-n(p+1)/2}x$ and $\tau = L^{-n}t$. Since $u_n(y,\tau)$ is the unique solution to IVP (27) in B_{f_n}, then $u(x,t)$ is the unique solution to IVP (3) in B. To prove the semi-group property, it is enough to apply Lemma 3.1 and (29), inductively. ∎

The previous results are concatenated in Theorem 3.1.

Theorem 3.1 *Under the hypotheses of Lemma 3.6, there is a constant A such that*

$$\lim_{n\to\infty} \left\| L^{n(p+1)/2} u(L^{n(p+1)/2}\cdot, L^n) - A f_p^*(\cdot) \right\| = 0.$$

Proof: Since $\|f_0\| < \overline{\varepsilon}$, it follows from Corollary 3.1 that the IVP (3) has a unique solution $u \in B$. It follows from the semi-group property and Lemma 3.4 that $f_n = A_n R^0_{L^n} f_p^* + g_n = L^{n(p+1)/2} u(L^{n(p+1)/2} x, L^n)$. Therefore, estimate (56) can be written as $\|f_n - A_n R^0_{L^n} f_p^*\| \le L^{n(\delta-1)/2}\|f_0\|$. Since $C_n \le K$ and $\|f_0\| < \overline{\varepsilon}$, using Lemma 3.5 and estimates (45) and (55), we obtain, for all $n = 1, 2, 3, \ldots$,

$$|A_{n+1} - A_n| < \frac{L^{n[p+3-\alpha(p+1)]/2}}{2L^{(1-\delta)/2}} \|f_0\|$$

and therefore, (A_n) is a Cauchy sequence in $\mathbb{R}$. Let $A \in \mathbb{R}$ be the limit of this sequence. Using the triangle inequality, $\|L^{n(p+1)/2} u(L^{n(p+1)/2} x, L^n) - A f_p^*\| \le \|L^{n(p+1)/2} u(L^{n(p+1)/2} x, L^n) - A_n R^0_{L^n} f_p^*\| + |A| \|R^0_{L^n} f_p^* - f_p^*\| + |A_n - A| \|R^0_{L^n} f_p^*\|$, which, from Lemma 3.2 and (42) can be upper bounded by

$$\frac{\|f_0\|}{L^{n(1-\delta)/2}} + |A| M \left| \frac{r(L^n)}{L^{n(p+1)}} \right| + \frac{L^{n[p+3-\alpha(p+1)]/2}}{2L_\delta^{(1-\delta)/2}\left(1 - L_\delta^{[p+3-\alpha(p+1)]/2}\right)} K_{p,q} \|f_0\|. \quad (59)$$

Then it is enough to take the limit when $n \to \infty$. ∎

Theorem 1.1 now follows from estimate (59) as we explain below.

Proof of Theorem 1.1: We have proved that (4) is valid when the initial data $f \equiv f_0$ is sufficiently small and $t = L^n$ $(n = 1, 2, \ldots)$, for $L > L_\delta$. In fact, it follows from (59) that, if $t = L^n$, then $\|\sqrt{t^{p+1}} u(\sqrt{t^{p+1}} x, t) - A f_p^*\| \le t^{(\delta-1)/2}\|f_0\| + |A| M \left| t^{-(p+1)} r(t) \right| + t^{[p+3-\alpha(p+1)]/2} L_\delta^{(\delta-1)/2} \left[2\left(1 - L_\delta^{[p+3-\alpha(p+1)]/2}\right)\right]^{-1} K_{p,q} \|f_0\|$. We can extend this bound to $t = \tau L^n$, with $\tau \in [1, L]$ and $L > L_\delta$ by replacing everywhere L by $\tau^{1/n} L$. Therefore, since the constants in (59) do not depend on the particular value of $L > L_\delta$ considered, the inequality above holds for all $t > L_\delta$. Taking the limit $t \to \infty$ finishes the proof. ∎

To conclude, we remark that we do not have an explicit expression for the limit A

of the sequence A_n. However, it should be clear that

$$A = \lim_{t\to\infty} \int u(x,t)\, \mathrm{d}x.$$

Also, we point out that the RG approach has been used to obtain higher order corrections to the asymptotic behavior, see [5, 8]).

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